Definition of Derivative

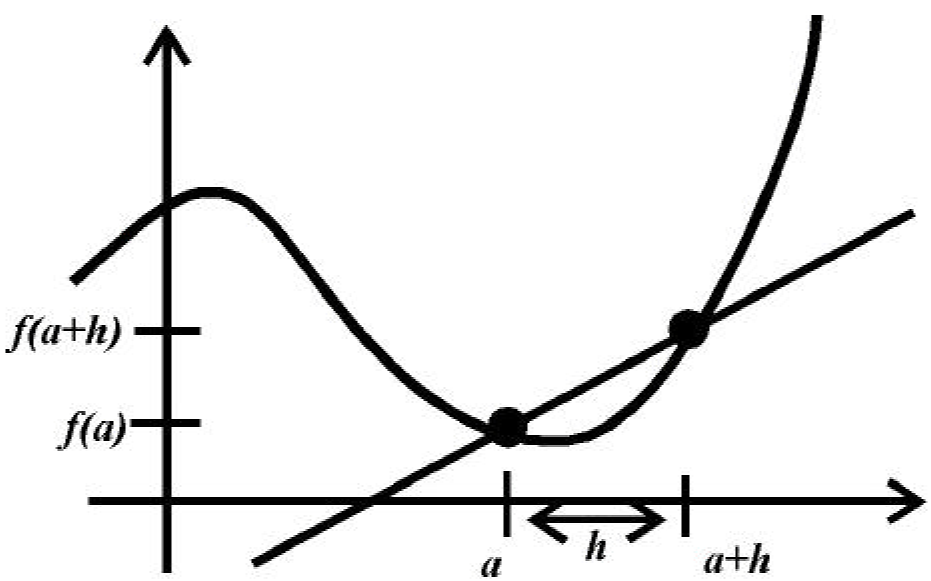
Concept

The most important fact to remember about the derivative is that it gives you the slope of a tangent line. As with most calculus quantities, the derivative is calculated as a **limit** of approximations, in this case the limit of slopes of secant lines. You can remember the definition as a precise translation of the general statement

The derivative is a limit of slopes of secant lines.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Let's look at this in some detail. For the curve $y=f(x)$, a secant line at $x=a$ is a line connecting the point on the curve at $x=a$ with another point on the curve. The figure below shows a curve with a secant line drawn in. The point at $x=a$ has y-value $f(a)$ since it is on the curve $y=f(x)$. We name the x-coordinate of the second point $a+h$, where h represents a (small) change in x. The corresponding y-value is $f(a+h)$. Recall that the slope of a line equals change in y over change in x. The change in y equals $f(a+h)–f(a)$ and the change in x equals $(a+h)–a = h$. Thus, the fraction in the definition equals the slope of the secant line. As h gets smaller, the two points of the secant line get closer together and the secant line approaches the tangent line.



Technique

The most common difficulty in using the definition comes in evaluating $f(a+h)$. Once you understand this step, you just need to carefully complete several steps of basic algebra.

Find the derivative at a=2 of $y=x^2+2x+3$.
Substituting in a=2, the definition of the derivative becomes

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

We need to evaluate each term in the numerator and then simplify the fraction. The hard part is $f(2+h)$. Be very exact about this: replace **each** occurrence of x in the formula for $f(x)$ with the entire expression $(2+h)$. So, instead of

$$f(x) = x^2 + 2x + 3$$

you should have

$$f(2+h) = (2+h)^2 + 2(2+h) + 3$$

Don't get in too much of a hurry. Write this down; your professor will appreciate your attention to detail. Now for the next step: evaluate $f(2)$. This is easy: just plug in:

$$f(2) = 2^2 + 2(2) + 3 = 4 + 4 + 3 = 11$$

Now, go back to the definition.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2 + 2(2+h) + 3 - 11}{h}$$

All that's left is to carefully simplify the fraction and substitute in h=0. Recall that there's a quick way to square a binomial like $(2+h)^2$. Square the first term, add 2 times the product of the terms and add the square of the last term (or, use the FOIL method). You should get $(2+h)^2 = 4 + 4h + h^2$. Also, $2(2+h) = 4 + 2h$. This gives us

$$f'(2) = \lim_{h\to 0} \frac{4 + 4h + h^2 + 4 + 2h + 3 - 11}{h} = \lim_{h\to 0} \frac{6h + h^2}{h}$$

It is important to notice that the numerical terms completely zeroed out. If the derivative exists, this will **definitely** happen. You can use it as a check on your work to this stage.

The reason that this is so important is that all remaining terms have a factor *h*. We can factor out *h* in the numerator, divide out the denominator and finally set *h*=0.

$$f\,'(2) = \lim_{h \to 0} \frac{6h + h^2}{h} = \lim_{h \to 0} \frac{h(6+h)}{h} = \lim_{h \to 0} (6+h) = 6$$

The keys here are to correctly evaluate *f*(2+*h*) and then slowly substitute into and simplify the fraction.

Extension

Several sections in Chapter 2 are devoted to deriving formulas so that you can avoid using the limit definition of the derivative. However, sometimes these formulas are not of practical use. For example, you may have measurements of several function values without having an exact formula for *f*(*x*). In this case, your best option is to approximate the derivative. You can do this by evaluating the slope of a secant line. For example, if you know that *f*(3)=5 and *f*(2)=1, an approximation of the derivative *f* '(2)is (5–1)/(3–2) = 4. If you know that *f*(2.2)=1.6 and *f*(2)=1, an approximation of *f* '(2) is (1.6–1)/(2.2–2)=3.

In applications, the derivative *f* '(*x*) is the **instantaneous rate of change** of the quantity *f*. Approximations like those shown above are **average rates of change** of the quantity *f*.