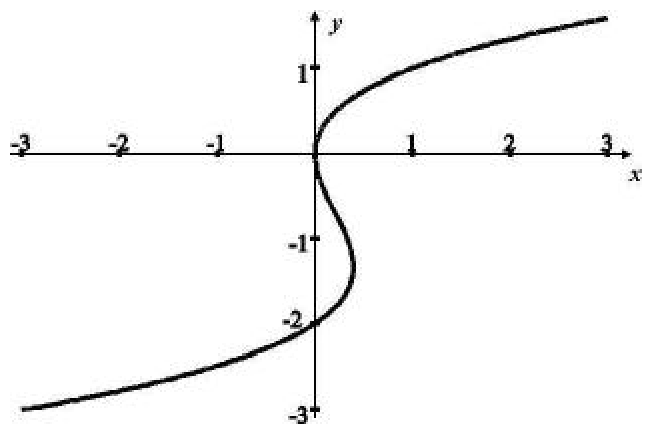
Implicit Differentiation

Concept

There are two main ideas involved in implicit differentiation. First, "implicit" refers to the situation where you can't solve a defining equation for *y* as a function of *x*. For example, the equation $y^3 + 2y^2 = 3x$ defines a curve (shown in the figure) implicitly because we can't solve the equation for *y*. Worse yet, the equation $y^3 + 2y^2 = 3x$ doesn't even define a function (the curve fails the vertical line test). To find the slope of a tangent line to this curve, we must use implicit differentiation since there is no function $y = f(x)$ to differentiate.



The second main idea in implicit differentiation is that we use the chain rule in a general way. You should be very comfortable finding the derivative of functions such as $(x^2+1)^3$ and $(\cos x+4)^3$. In both cases, you have a function of the form $(f(x))^3$ and the derivative has the form $3\,(f(x))^2 f\,'(x)$. Replacing $f(x)$ with *y*, the derivative of y^3 is $3\,y^2 y'$. Be sure you understand why the y' appears in this derivative. (It's the chain rule applied to the **function** *y*.) This type of calculation is the key to using implicit differentiation. Technically, this is legal only if *y* is a function of *x*. For this to be true, you may need to restrict your attention to a small portion of the curve which passes the vertical line test.

Technique

Find an equation of the tangent line to the curve $y^3 + 2y^2 = 3x$ at the point (1,1).

The first step is to take the derivative of each term in the equation. Thinking of *y* as a function $f(x)$, the derivative of both terms on the left side of the equation will involve chain rules as discussed above.

$$y^3 \quad + \quad 2y^2 = 3x$$
$$3\,y^2 y' \quad + \quad 4\,y\,y' = 3$$

If you understand this first step, you only need to do some algebra to find the derivative. The remaining steps will be the same for any implicit differentiation problem. You need to put all terms involving *y*' on one side of the equation and all other terms on the other side (no work needed in this example). Then factor out *y*', divide by the factor of *y*' and finally substitute in the values for *x* and *y*. In this case, we have

$$(3y^2+4y)\,y' = 3$$

$$y' = \frac{3}{3y^2+4y}$$

$$y' = \frac{3}{3+4}$$

The slope of the tangent line *y*' equals 3/7. Using the point (1,1), an equation of the tangent line is $y = 3/7\,(x-1) + 1$.

Extension

The above example is simple in that the equation of the curve doesn't have any terms involving both *x* and *y*. If there are such terms, you need to use the product, quotient and chain rules as necessary. For example, suppose one term in your equation is x^2y^2. This is the **product** of x^2 and y^2, so you should use the product rule. Remember that the derivative of y^2 will require the chain rule as discussed above.

$$(x^2y^2)'$$
$$(x^2)'\,y^2 + x^2\,(y^2)'$$
$$2x\,y^2 + x^2\,2y\,y'$$

From the first line to the second line, we wrote out the form of the product rule. From the second line to the third line, we found the indicated derivatives. Writing out each step can be helpful as you get used to these manipulations.