Riemann sums

Concept

The concept of a Riemann sum is simple: you add up the areas of a number of rectangles. In the problems you will work in this chapter, the width of each rectangle (called Δx) is the same. The heights of the rectangles vary according to the values $f(x_i)$ of a given function at different points. The area of the rectangle equals height times width, or $f(x_i)\ \Delta x$. We denote the sum of n of these rectangles by

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

To compute a Riemann sum, you need to identify (1) the function $f(x)$, (2) the value of Δx and (3) the points at which to evaluate the function. Then follow the pattern of the examples below. As with many calculus problems, these are multi-step problems, so you should work enough problems to become comfortable with the sequence of steps involved.

Technique

We will illustrate two types of Riemann sum problems, one where we compute a specific Riemann sum and one where we compute a definite integral as a limit of Riemann sums.

Compute a Riemann sum of $f(x)=x^2+2$ on the interval [1,3] using n=4 rectangles and midpoint evaluation.

The function is given to us. The interval has length 2 and we divide it into 4 pieces, so the length of one subinterval is $\Delta x = 2/4 = 0.5$. We need to determine the 4 points at which to evaluate $f(x)$. First, divide the interval [1,3] into 4 pieces, then find the midpoint of each subinterval. This is shown below.

```
[----------------]        [----x----x----x----]      [--m--x--m--x--m--x--m--]
1                3        1   1.5   2   2.5   3      1     1.5     2     2.5     3
                                                        1.25   1.75   2.25   2.75
```

The evaluation points are x = 1.25, 1.75, 2.25 and 2.75. With Δx=0.5, the Riemann sum is given by

$$f(1.25)\ 0.5 + f(1.75)\ 0.5 + f(2.25)\ 0.5 + f(2.75)\ 0.5$$
$$[\ f(1.25) + f(1.75) + f(2.25) + f(2.75)\ ]\ 0.5$$
$$[\ 3.5625 + 5.0625 + 7.0625 + 9.5625\ ]\ 0.5$$
$$12.625$$

Use Riemann sums to compute the definite integral

$$\int_0^2 (x+1)\,dx$$

For this type of calculation, it is usually easiest to use right-endpoint evaluation. If we divide the interval [0,2] into *n* pieces, we have $\Delta x = 2/n$, with endpoints 0, 2/*n*, 4/*n*, 6/*n*, and so on. Thinking of 4/*n* as 2 (2/*n*) and 6/*n* as 3 (2/*n*), we have the general evaluation point $x_I = i\,(2/n) = 2i/n$. The function in this problem is $f(x) = x+1$. Replacing *x* with our formula for the evaluation points, we have function values of

$$f(x_{Ii}) = f(2i/n) = 2i/n + 1$$

Putting this together with $\Delta x = 2/n$, the general Riemann sum looks like

$$\sum_{i=1}^{n}\left(\frac{2i}{n}+1\right)\frac{2}{n} = \sum_{i=1}^{n}\left(\frac{4i}{n^2}+\frac{2}{n}\right) = \frac{4}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}\sum_{i=1}^{n} 1$$

To evaluate the sum, use the summation formulas given in Theorem 2.1 of section 4.2.

$$\frac{4}{n^2}\frac{n(n+1)}{2} + \frac{2}{n}n = \frac{2(n+1)}{n} + 2$$

Finally, take the limit of this expression as *n* goes to ∞.

$$\lim_{n\to\infty}\frac{2(n+1)}{n} + 2 = 2 + 2 = 4$$

The integral equals 4.

Extension

The two types of examples given above are related to each other. Since the integral equals a limit of Riemann sums, any specific Riemann sum gives an approximation of an integral. In the first example above, 12.625 is an approximation of the integral

$$\int_1^3 (x^2+2)\,dx$$

It can be shown that the exact value of this integral is 38/3, so the approximation is not bad. To obtain a better approximation, you could use the same midpoint evaluation with a larger number of rectangles or you could use a different method altogether. The midpoint evaluation method is called the **midpoint rule**. Midpoint evaluation tends to be more accurate than right-endpoint or left-endpoint evaluation. However, a different approximation could be obtained by computing right-endpoint and left-endpoint evaluations and averaging the two. This is called the **trapezoid rule** and is discussed in some detail in section 4.7. A very accurate method called **Simpson's rule** is also developed in that section.